ISRAEL JOURNAL OF MATHEMATICS 108 (1998), 145–171

CONCERNING BOURGAIN'S ℓ_1-INDEX OF A BANACH SPACE*

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ABSTRACT

A well known argument of James yields that if a Banach space X contains ℓ_1^n's uniformly, then X contains ℓ_1^n's almost isometrically. In the first half of the paper we extend this idea to the ordinal ℓ_1-indices of Bourgain. In the second half we use our results to calculate the ℓ_1-index of certain Banach spaces. Furthermore we show that the ℓ_1-index of a separable Banach space not containing ℓ_1 must be of the form ω^α for some countable ordinal α.

1. Introduction

It is well known that if ℓ_p $(1 \leq p < \infty)$ or c_0 is crudely finitely representable in a Banach space X, then it is finitely representable in X. This was shown for ℓ_1 and c_0 by R.C. James [J] and for ℓ_p $(1 < p < \infty)$ it is a consequence of Krivine's theorem [K] as noted by Rosenthal [R], [L]. We may state this as:

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* Research supported by the NSF and TARP.
Received August 8, 1996

For all $p \in [1,\infty]$, every $K \geq 1$, each $m \geq 1$, and every $\varepsilon > 0$, there exists n such that if $(x_i)_1^n$ is a normalized basic sequence in a Banach space X with $(x_i)_1^n \overset{K}{\sim}$ uvb ℓ_p^n, then there exists a normalized block basis $(y_i)_1^m$ of $(x_i)_1^n$ satisfying $(y_i)_1^m \overset{1+\varepsilon}{\sim}$ uvb ℓ_p^m.

Separable Banach spaces not containing ℓ_1 may differ in the complexity of ℓ_1^n's embedded inside. This complexity is measured in part by Bourgain's ℓ_1-index [B]. Bourgain considered trees $T(X,K)$ whose nodes are finite basic sequences in the unit ball of a Banach space X, K-equivalent to the unit vector basis of some finite dimensional ℓ_1, for a fixed K. The ℓ_1-K-ordinal index of X, $I(X,K)$, was then defined to be the supremum of the orders of such trees.

The definition of the ℓ_1-trees constructed by Bourgain may be extended to ℓ_p-trees $(1 < p \leq \infty)$ (we explain all the unfamiliar terms in the next section). We extend the results on finite representability of ℓ_p in X to ℓ_p-trees for $p = 1$ and ∞. We prove the following theorem in Section 4.

THEOREM 1.1: *For $p = 1$ or ∞, for each $K > 1$, for every $\alpha < \omega_1$, and any $\varepsilon > 0$, there exists $\beta < \omega_1$ such that for all Banach spaces X, if T is an ℓ_p-tree on X with constant K and order, $o(T) \geq \beta$, then there exists an ℓ_p block tree T' of T with constant $1+\varepsilon$ and order, $o(T') \geq \alpha$.*

This theorem is not true in general for $1 < p < \infty$, and in the final section we explain why not. We also show how the same ideas may be applied to the ℓ_1-$\mathcal{S}_\alpha$-spreading models introduced by Kiriakouli and Negrepontis [KN].

In Section 5 we apply our results to the problem of calculating Bourgain's ℓ_1-index $I(X)$ of certain spaces X. We show for example that if X is Tsirelson's space, then $I(X) = \omega^\omega$. We prove that $I(X)$ is always of the form ω^α and relate $I(X)$ to the "block" Bourgain ℓ_1-index $I_b(X)$ for spaces with a basis. Both indices are defined in Section 5.

2. Preliminaries on trees

By a **tree** we shall mean a countable, non-empty, partially ordered set $(T, \leq)$ for which the set $\{y \in T\colon y < x\}$ is linearly ordered and finite for each $x \in T$. The elements of T are called **nodes**. The **predecessor node** of x is the maximal element x' of the set $\{y \in T\colon y < x\}$, so that if $y < x$, then $y \leq x'$. The **initial nodes** of T are the minimal elements of T and the **terminal nodes** are the maximal elements. A **subtree** of a tree T is a subset of T with the induced ordering from T. This is clearly again a tree. Further, if $T' \subset T$ is a subtree of T and $x \in T$, then we write $x < T'$ to mean $x < y$ for every $y \in T'$. We will also consider trees related to some fixed set X. **A tree on a set** X is a partially ordered subset $T \subseteq \bigcup_{n=1}^{\infty} X^n$ such that $(x_1, \ldots, x_m) \leq (y_1, \ldots, y_n)$ implies $m \leq n$ and $x_i = y_i$ for $i = 1, \ldots, m$.

The property of trees which is most interesting here is their **order**. Before we can define this we must recall some terminology. Let the **derived tree** of a tree T be $D(T) = \{x \in T: x < y \text{ for some } y \in T\}$. It is easy to see that this is simply T with all of its terminal nodes removed. We then associate a new tree T^α to each ordinal α inductively as follows. Let $T^0 = T$, then given T^α let $T^{\alpha+1} = D(T^\alpha)$. If α is a limit ordinal, and we have defined T^β for all $\beta < \alpha$, let $T^\alpha = \bigcap_{\beta<\alpha} T^\beta$. A tree T is **well-founded** provided there exists no subset $S \subseteq T$ with S linearly ordered and infinite. The order of a well-founded tree T is defined as $o(T) = \inf\{\alpha: T^\alpha = \emptyset\}$.

A tree T on a topological space X is said to be **closed** provided the set $T \bigcap X^n$ is closed in X^n, endowed with the product topology, for each $n \geq 1$. We have the following result (see [B], [D]) concerning the order of a closed tree on a Polish space.

PROPOSITION 2.1: *If T is a well-founded, closed tree on a Polish (separable, complete, metrizable) space, then $o(T) < \omega_1$.*

A map $f: T \to T'$ between trees T and T' is a **tree isomorphism** if f is one to one, onto and an order isomorphism ($x < y$ if and only if $f(x) < f(y)$). We will write $T \simeq T'$ if T is tree isomorphic to T' and $f: T \xrightarrow{\sim} T'$ to denote a tree isomorphism. From now on we shall simply write **isomorphism** rather than tree isomorphism.

Definition 2.2: A tree t is a **minimal tree of order** α, for some ordinal $\alpha < \omega_1$, if for each tree T of order α there exists a subtree $T' \subset T$ of order α which is isomorphic to t. Notice that if t is a minimal tree of order α, then any subtree of t of order α is also a minimal tree of order α. We construct certain minimal trees for each ordinal $\alpha < \omega_1$ in Section 3.

If X is a Banach space and $(x_i)_1^m \subset X$ with $\|x_i\| = 1$ $(i = 1, \ldots, m)$ we write $(x_i)_1^m \overset{K}{\sim} \text{uvb } \ell_p^m$ if there exist constants c, C with $c^{-1}C \leq K$ and

$$c\Big(\sum_1^m |a_i|^p\Big)^{\frac{1}{p}} \leq \Big\|\sum_1^m a_i x_i\Big\| \leq C\Big(\sum_1^m |a_i|^p\Big)^{\frac{1}{p}}$$

for all $(a_i)_1^m \subset \mathbf{R}$.

Definition 2.3: An ℓ_p-K-**tree** on a Banach space X is a tree T on X such that $T \subseteq \bigcup_{n=1}^{\infty} S(X)^n$ and $(x_i)_1^m \overset{K}{\sim} \text{uvb } \ell_p^m$ for each $(x_1, \ldots, x_m) \in T$. We say T is an **ℓ_p-tree** on X if T is an ℓ_p-K-tree for some K. For $p = 1$ this definition is slightly different to that in [B] where an ℓ_1-K-tree is the largest tree of this form. In fact our trees are subtrees of those.

Definition 2.4: Let T be a tree on the unit sphere $S(X)$ of a Banach space X. We say S is a **block tree** of T, written $S \preceq T$, if S is a tree on $S(X)$ such that there exists a subtree $T' \subset T$ and an isomorphism f: $T' \overset{\sim}{\to} S$ satisfying: $f((x_i)_1^m) = (y_i)_1^n$ is a normalized block basis of $(x_i)_1^m$ for each $(x_i)_1^m \in T'$, and if $(x_i)_1^k \in T'$ for some $k < m$, then there exists $l < m$ such that $(y_i)_1^l = f((x_i)_1^k)$ and $(y_i)_{l+1}^n$ is a normalized block basis of $(x_i)_{k+1}^m$.

For each node $(y_1, \ldots, y_k) = f(x) \in S$ we call x the **parent node** of $(y_1, \ldots, y_k)$. Note that if T is an ℓ_1-K-tree on X and S is a block tree of T, then S is also an ℓ_1-K-tree on X. However, if T is an ℓ_p-K-tree on X for $p > 1$ and S is a block tree of T, then S is at worst an ℓ_p-K^2-tree on X.

A **block** of a tree T as above is simply a normalized vector v in the linear span of some node $(x_1, \ldots, x_n)$ of T.

3. Ordinal trees

Most of the work needed to prove Theorem 1.1 is concerned with constructing certain general trees consisting of collections of finite subsets of ordinals ordered by inclusion. We first construct specific minimal trees T_α of order α for every ordinal $\alpha < \omega_1$. Once this is done we construct "replacement trees" $T(\alpha, \beta)$ which are formed by replacing each node of T_α by one or more copies of T_β, and show that $T(\alpha, \beta)$ is a minimal tree of order $\beta \cdot \alpha$. This gives us in some sense a "tree within a tree" or "an α tree of β trees".

These two results are used as follows: Given an arbitrary ℓ_1-K-tree on X with $o(T) \geq \alpha^2$ we can find a subtree isomorphic to $T(\alpha, \alpha)$. For one of the α trees inside this we either have a good constant—in which case we are finished—or we take a vector in the linear span of one of its nodes with a bad constant. Putting some of these vectors together yields a block tree of order α, each of whose nodes is "bad", and then following the original argument of James these vectors together have a good constant.

We now define the trees T_α, and prove in Lemma 3.3 that T_α is minimal of order α.

Definition 3.1: We define **minimal trees** T_α of order α for each countable ordinal α as subsets of $[1, \gamma]^{<\omega}$ ordered by inclusion, for some ordinal $\gamma = \gamma(\alpha) < \omega_1$ where, if S is any set, then $[S]^{<\omega}$ is the collection of all finite subsets of S. We choose $\gamma(\alpha)$ and T_α by induction as follows: Let $T_1 = \{\{1\}\}$. Given $T_\alpha \subset [1,\gamma]^{<\omega}$ for some ordinal $\gamma < \omega_1$, let

$$T_{\alpha+1} = \{a \cup \{\gamma+1\} \colon a \in T_\alpha\} \cup \{\{\gamma+1\}\}.$$

Note that for $\beta < \alpha$,

$$(T_{\alpha+1})^\beta = \{a \cup \{\gamma+1\} \colon a \in (T_\alpha)^\beta\} \cup \{\{\gamma+1\}\} \text{ and } (T_{\alpha+1})^\alpha = \{\{\gamma+1\}\}.$$

Thus $o(T_{\alpha+1}) = \alpha+1$ as required.

Finally, to define T_α for α a limit ordinal, let $\alpha_n \nearrow \alpha$ be a sequence of ordinals increasing to α, and let $T_{\alpha_n} \subset [1,\beta_n]^{<\omega}$ for some $\beta_n < \omega_1$. Let $\beta = \sup_n \beta_n$ and $\gamma_n = \beta + n$ for each n. Let $\tilde{T}_{\alpha_n} = \{a \cup \{\gamma_n\} \colon a \in T_{\alpha_n}\}$ and let $T_\alpha = \bigcup_1^\infty \tilde{T}_{\alpha_n}$, ordered by inclusion. Notice that $\tilde{T}_{\alpha_n}$ is the same tree as T_{α_n} with the same order and structure, but the nodes have simply been relabeled. The reason for doing this is that nodes from different trees are now incomparable, and so the union $\bigcup_1^\infty \tilde{T}_{\alpha_n}$ is a disjoint union.

To give an idea of what these trees look like we will construct the trees T_n and T_ω explicitly.

$$\begin{aligned}
T_1 &= \{\{1\}\}\\
T_2 &= \{\{1,2\},\{2\}\}\\
T_3 &= \{\{1,2,3\},\{2,3\},\{3\}\}\\
&\vdots\\
T_n &= \{\{1,2,3,\ldots,n\},\{2,3,\ldots,n\},\ldots,\{n-1,n\},\{n\}\}.
\end{aligned}$$

Then to construct T_ω we use the trees $\tilde{T}_n$ $(n \geq 1)$ as described above.

$$\begin{aligned}
\tilde{T}_1 &= \{\{1,\omega+1\}\}\\
\tilde{T}_2 &= \{\{1,2,\omega+2\},\{2,\omega+2\}\}\\
&\vdots\\
\tilde{T}_n &= \{\{1,\ldots,n,\omega+n\},\ldots,\{n,\omega+n\}\}\\
T_\omega &= \{\{1,\omega+1\},\{1,2,\omega+2\},\{2,\omega+2\},\ldots,\{1,\ldots,n,\omega+n\},\\
&\qquad \ldots,\{n,\omega+n\},\ldots\}.
\end{aligned}$$

LEMMA 3.2: *Let $\alpha < \omega_1$ be a limit ordinal and let T be a countable tree of order α. Then there exist a sequence (α_n) of successor ordinals and a sequence (t_n) of subtrees $t_n \subset T$ with $\alpha = \sup_n \alpha_n$, $o(t_n) = \alpha_n$ and $T = \bigcup_1^\infty t_n$. Moreover* *the trees (t_n) are mutually incomparable, i.e., if $x \in t_n$ and $y \in t_m$ with $n \neq m$, then x and y are incomparable.*

Proof: Suppose that T has only finitely many initial nodes; let these be $x_1, \ldots, x_n$, and let $t_i = \{y \in T\colon y \geq x_i\}$. Then $\alpha = o(T) = \max_{1 \leq i \leq n} o(t_i) = o(t_{i_0})$ for some $i_0 \leq n$. Let $t = \{y \in t_{i_0}\colon y > x_{i_0}\}$ and let $\beta = o(t)$. Since $\{x_{i_0}\}$ is the unique initial node of t_{i_0}, it follows that $t_{i_0} = t \cup \{x_{i_0}\}$ and hence $(t_{i_0})^\beta = \{x_{i_0}\}$. Thus $\alpha = o(t_{i_0}) = \beta + 1$, a successor ordinal, contradicting the assumption that α is a limit ordinal.

Thus T must have infinitely many initial nodes; let these be $(x_n)_1^\infty$ and let $t_n = \{y \in T\colon y \geq x_n\}$, $\alpha_n = o(t_n)$. Note that these trees are mutually incomparable since the nodes $(x_n)_1^\infty$ are incomparable. We find that α_n is a successor ordinal using the same argument as above and from the definition of the order of a tree we have that $o(T) = \sup_n o(t_n)$ and hence $\alpha = \sup_n \alpha_n$. ∎

Lemma 3.3: *T_α is a minimal tree of order α.*

Proof: The order of T_α is clear from the construction; we prove here that if T is any tree of order $\alpha < \omega_1$, then there exists a subtree $T' \subset T$ such that T' is isomorphic to T_α. We use induction on α, the order of T. The result is obvious for $\alpha = 1$.

Suppose the lemma is true for the ordinal $\alpha < \omega_1$. Let T have order $\alpha + 1$, and hence $T^\alpha \neq \emptyset$. Let x be a terminal node of T^α and let $\tilde{T} = \{y \in T\colon y > x\}$; then $o(\tilde{T}) = \alpha$. By assumption there exists a subtree $\tilde{T}'$ of $\tilde{T}$ and an isomorphism $f\colon T_\alpha \stackrel{\sim}{\to} \tilde{T}'$. Clearly $T' = \tilde{T}' \cup \{x\}$ is a subtree of T of order $\alpha + 1$ and we can extend f to $F\colon T_{\alpha+1} \stackrel{\sim}{\to} T'$ to show that T' is isomorphic to $T_{\alpha+1}$ as follows. Recall from Definition 3.1 that we obtain $T_{\alpha+1}$ from T_α by setting $T_{\alpha+1} = \{a \cup \{\gamma+1\}\colon a \in T_\alpha\} \cup \{\{\gamma+1\}\}$. Setting $F(\{\gamma+1\}) = x$ and $F(a \cup \{\gamma+1\}) = f(a)$ makes F the required isomorphism.

If α is a limit ordinal, let the lemma be true for all $\beta < \alpha$ and let T have order α. By Lemma 3.2, $T = \bigcup_1^\infty t_n$ where $o(t_n) = \beta_n$, $\alpha = \sup_n \beta_n$, each β_n is a successor ordinal, and the trees (t_n) are mutually incomparable. Let $\alpha_n \nearrow \alpha$ be the sequence of ordinals increasing to α, and let $\tilde{T}_{\alpha_n}$ be the trees, from the definition of the minimal tree T_α, Definition 3.1. Let (β_{r_n}) be a subsequence of (β_n) so that $\alpha_n \leq \beta_{r_n}$ for all n. Each tree t_{r_n} contains a subtree of order α_n; hence, by assumption, for each n there exists $t'_{r_n} \subset t_{r_n}$ and an isomorphism $f_n\colon T_{\alpha_n} \stackrel{\sim}{\to} t'_{r_n}$. Using the notation of Definition 3.1 we define $\tilde{f}_n\colon \tilde{T}_{\alpha_n} \stackrel{\sim}{\to} t'_{r_n}$ by $\tilde{f}_n(a \cup \{\gamma_n\}) = f_n(a)$. Let $T' = \bigcup_1^\infty t'_{r_n}$ and $f\colon T_\alpha \stackrel{\sim}{\to} T'$ be the function $f = \bigcup_1^\infty \tilde{f}_n$. ∎

Remark 3.4: It follows that if T is a tree of order $\beta \geq \alpha$, then there exists a subtree $T' \subset T$ such that T' is isomorphic to T_α.

We now construct the replacement trees $T(\alpha,\beta)$, for each pair of ordinals α, $\beta < \omega_1$, promised earlier. First we construct the trees by induction, then we prove that $T(\alpha,\beta)$ has order $\beta \cdot \alpha$. Finally we show that $T(\alpha,\beta)$ is isomorphic to a subtree of $T_{\beta\cdot\alpha}$ and hence is a minimal tree of order $\beta\cdot\alpha$ as required. The key to all of these proofs is to use induction on α for an arbitrary β.

Definition 3.5: For each pair $\alpha,\beta < \omega_1$ we construct a tree $T(\alpha,\beta)$ which we call a **replacement tree** and a map $f_{\alpha,\beta}\colon T(\alpha,\beta) \to T_\alpha$ satisfying:

(i) For each $x \in T_\alpha$ there exists $I = \{1\}$ or $\mathbf{N}$ and trees $t(x,j) \simeq T_\beta$, $j \in I$, such that $f^{-1}_{\alpha,\beta}(x) = \bigcup_{j\in I} t(x,j)$ (incomparable union) with $I = \{1\}$ if α is a successor ordinal and x is the unique initial node, or $\beta < \omega$, and $I = \mathbf{N}$ otherwise.

(ii) For each pair $a, b \in T(\alpha,\beta)$, $a \leq b$ implies $f_{\alpha,\beta}(a) \leq f_{\alpha,\beta}(b)$.

For each $\beta < \omega_1$, let $T(1,\beta) = T_\beta$ and $f_{1,\beta}\colon T(1,\beta) \to T_1$ be given by $f_{1,\beta}(a) = \{1\}$ $\forall a \in T(1,\beta)$. Let $\alpha < \omega_1$ and suppose we have defined $T(\alpha,\beta)$ and $f_{\alpha,\beta}$ for each $\beta < \omega_1$. Roughly speaking, what we do to go from α to $\alpha+1$ is to take T_β and then after each of its terminal nodes we put a $T(\alpha,\beta)$ tree. This will give us the required tree, but we have to ensure that it is well defined and that we keep track of the order relation.

Recall from Definition 3.1 that $T_{\alpha+1} = \{a\cup\{\gamma+1\}\colon a \in T_\alpha\}\cup\{\{\gamma+1\}\}$ for some $\gamma < \omega_1$. Let δ_1, δ_2 be countable ordinals with $T(\alpha,\beta) \subset [1,\delta_1]^{<\omega}, T_\beta \subset [1,\delta_2]^{<\omega}$. Define a map $\tilde{}\colon [1,\delta_2] \to [\delta_1+1, \delta_1+\delta_2]$ by $\eta \mapsto \tilde{\eta} = \delta_1 + \eta$. For all ordinals λ,μ,ν, we have $\lambda+\mu = \lambda+\nu \Rightarrow \mu = \nu$, hence this map is one to one. Thus, if we define $\tilde{a} = \{\tilde{\eta}\colon \eta \in a\}$ for $a \in T_\beta$ and $\tilde{T}_\beta = \{\tilde{a}\colon a \in T_\beta\}$, then $\tilde{T}_\beta \simeq T_\beta$ as the map $\tilde{}$ is merely relabeling the nodes, but the trees $\tilde{T}_\beta$ and $T(\alpha,\beta)$ are now incomparable since if $a \in \tilde{T}_\beta$, $b \in T(\alpha,\beta)$, then $a \cap b = \emptyset$.

Let $(\tilde{x}_n)_I$ ($I = \{1\}$ or $\mathbf{N}$) be the set of terminal nodes, of $\tilde{T}_\beta$, a sequence of incomparable nodes, and let

$$T(\alpha+1,\beta) = \bigcup_{n\in I}\{a \cup \tilde{x}_n\colon a \in T(\alpha,\beta)\} \cup \tilde{T}_\beta,$$

$$f_{\alpha+1,\beta}(x) = \begin{cases} f_{\alpha,\beta}(a) \cup \{\gamma+1\}, & x = a \cup \tilde{x}_n \quad \text{for some } a \in T(\alpha,\beta), \\ \{\gamma+1\}, & x \in \tilde{T}_\beta. \end{cases}$$

We need to show that the map $f_{\alpha+1,\beta}$ satisfies the required properties. Let $y \in T_{\alpha+1}$. If $y = \{\gamma+1\}$, then $f^{-1}_{\alpha+1,\beta}(y) = \tilde{T}_\beta \simeq T_\beta$. Otherwise $y = a \cup \{\gamma+1\}$

for some $a \in T_\alpha$ and hence

$$f^{-1}_{\alpha+1,\beta}(y) = \bigcup_{n\in I}\{b \cup \tilde{x}_n \colon b \in f^{-1}_{\alpha,\beta}(a)\}$$
$$= \bigcup_{n\in I}\bigcup_{i\in I'} t_{n,i} \quad \text{where } t_{n,i} \simeq T_\beta \text{ and } I' = \{1\} \text{ or } \mathbf{N}$$
$$= \bigcup_{j\in I''} t(y,j) \quad \text{where } t(y,j) \simeq T_\beta \text{ and } I'' = \{1\} \text{ or } \mathbf{N}$$

as required. Furthermore, the $t(y,j)$'s are incomparable. The second property is clear.

If α is a limit ordinal, let $\alpha_n \nearrow \alpha$ be the sequence of ordinals increasing to α from Definition 3.1 and suppose we have constructed $T(\alpha_n,\beta)$, $f_{\alpha_n,\beta}$ for each α_n. Let $T(\alpha_n,\beta) \subset [1,\delta_n]^{<\omega}$, $\delta = \sup_n \delta_n < \omega_1$, and set $\gamma_n = \delta + n$ for each n. Then, as in the definition of the minimal trees, let $\tilde{T}(\alpha_n,\beta) = \{a \cup \{\gamma_n\} \colon a \in T(\alpha_n,\beta)\}$, $\tilde{f}_{\alpha_n,\beta}(a \cup \{\gamma_n\}) = f_{\alpha_n,\beta}(a)$, and let $T(\alpha,\beta) = \bigcup_1^\infty \tilde{T}(\alpha_n,\beta)$, $f_{\alpha,\beta} = \bigcup_1^\infty \tilde{f}_{\alpha_n,\beta}$.

LEMMA 3.6: $o(T(\alpha,\beta)) = \beta\cdot\alpha$.

Proof: We proceed by induction on α for an arbitrary fixed β. The result is obvious for $\alpha = 1$.

Suppose $o(T(\alpha,\beta)) = \beta\cdot\alpha$. By the construction of $T(\alpha+1,\beta)$ we have that $(T(\alpha+1,\beta))^{\beta\cdot\alpha} = \tilde{T}_\beta$ and hence $o(T(\alpha+1,\beta)) = \beta\cdot\alpha+\beta = \beta\cdot(\alpha+1)$. If α is a limit ordinal and $o(\tilde{T}(\alpha_n,\beta)) = o(T(\alpha_n,\beta)) = \beta\cdot\alpha_n$ for each n, where $T(\alpha,\beta) = \bigcup_1^\infty \tilde{T}(\alpha_n,\beta)$ from Definition 3.5, then $o(T(\alpha,\beta)) = \sup_n o(\tilde{T}(\alpha_n,\beta)) = \sup_n \beta\cdot\alpha_n = \beta\cdot\alpha$. ∎

The last of our results on these specially defined trees is the following:

LEMMA 3.7: *$T(\alpha,\beta)$ is a minimal tree of order $\beta\cdot\alpha$.*

Proof: Since $o(T(\alpha,\beta)) = \beta\cdot\alpha$ and $T_{\beta\cdot\alpha}$ is a minimal tree of order $\beta\cdot\alpha$, then by Remark 3.4 it is sufficient to prove that $T(\alpha,\beta)$ is isomorphic to a subtree of $T_{\beta\cdot\alpha}$. We prove this by induction on α for an arbitrary β. The result is obvious for $\alpha = 1$ since $T(1,\beta) = T_\beta$.

Suppose $T(\alpha,\beta)$ is isomorphic to a subtree of $T_{\beta\cdot\alpha}$ and hence is a minimal tree of order $\beta\cdot\alpha$. Now, $o(T_{\beta\cdot(\alpha+1)}) = \beta\cdot(\alpha+1)$ so $o((T_{\beta\cdot(\alpha+1)})^{\beta\cdot\alpha}) = \beta$, thus since T_β is minimal it is isomorphic to a subtree of $(T_{\beta\cdot(\alpha+1)})^{\beta\cdot\alpha}$. But by construction $(T(\alpha+1,\beta))^{\beta\cdot\alpha} \simeq T_\beta$ and hence is isomorphic to a subtree t_0 of $(T_{\beta\cdot(\alpha+1)})^{\beta\cdot\alpha}$. Let the isomorphism which sends $(T(\alpha+1,\beta))^{\beta\cdot\alpha}$ onto $t_0 \subseteq (T_{\beta\cdot(\alpha+1)})^{\beta\cdot\alpha}$ be $a \mapsto a'$, so that if $(x_n)_1^\infty$ are the terminal nodes of $(T(\alpha+1,\beta))^{\beta\cdot\alpha}$, then $(x'_n)_1^\infty$ are their images in $(T_{\beta\cdot(\alpha+1)})^{\beta\cdot\alpha}$, the terminal nodes of t_0, under this map. Let

$T(x'_n) = \{y \in T_{\beta\cdot(\alpha+1)}: y > x'_n\} \subset T_{\beta\cdot(\alpha+1)}$, then $o(T(x'_n)) \geq \beta \cdot \alpha$ for each n. Now, by assumption, for each n there exists a subtree t_n of $T(x'_n)$ isomorphic to $T(\alpha, \beta)$ and hence the subtree $\tilde{T} = (\bigcup_1^\infty t_n) \cup t_0$ of $T_{\beta\cdot(\alpha+1)}$ is isomorphic to $T(\alpha+1, \beta)$ as required.

Let α be a limit ordinal with $T(\alpha, \beta) = \bigcup_1^\infty \tilde{T}(\alpha_n, \beta)$ via the construction in Definition 3.5, and let $T(\alpha_n, \beta)$ be isomorphic to a subtree of $T_{\beta\cdot\alpha_n}$ for each n. Then $\tilde{T}(\alpha_n, \beta)$ is isomorphic to a subtree of $\tilde{T}_{\beta\cdot\alpha_n}$ for all n, where $\tilde{T}_{\beta\cdot\alpha_n} = \{a \cup \{\gamma_n\}: a \in T_{\beta\cdot\alpha_n}\}$ for some γ_n, from Definition 3.1, and hence $T(\alpha, \beta)$ is isomorphic to a subtree of $T_{\beta\cdot\alpha} = \bigcup_1^\infty \tilde{T}_{\beta\cdot\alpha_n}$ as required. ∎

4. Proof of Theorem 1.1

We have shown everything we need about trees on subsets of ordinals and we now want to apply this to ℓ_1-trees on a Banach space X.

Definition 4.1: Let T be a tree on a set X and let T' be a subtree of T. We define another tree on X, the **restricted subtree $R(T')$ of T' with respect to T**. Let $x = (x_i)_1^n \in T'$ and let y be the unique initial node of T' such that $y \leq x$; let $m \leq n$ be such that $y = (x_i)_1^m$. If y is also an initial node of T, then set $k = 0$, otherwise let $k < m$ be such that $(x_i)_1^k$ is the predecessor node of y in T. Finally, setting $R(x) = (x_{k+1}, \ldots, x_n)$, we define $R(T') = \{R(x): x \in T'\}$. It is easy to see that $R(T')$ is isomorphic to T'.

LEMMA 4.2: *Let T be a tree on a Banach space X of order $\beta \cdot \alpha$ isomorphic to $T(\alpha, \beta)$, and let $F: T \to T_\alpha$ be the map from Definition 3.5 satisfying: for all $x \in T_\alpha$, $F^{-1}(x) = \bigcup_I T_n(x)$, where $I = \{1\}$ or $\mathbf{N}$, $T_n(x) \simeq T_\beta$ and the $T_n(x)$'s are mutually incomparable. For each $x \in T_\alpha$ and $n \geq 1$, let $b(x, n)$ be a block of $R(T_n(x))$. Then there exists a block tree T' of T and an isomorphism $g: T' \xrightarrow{\sim} T_\alpha$ satisfying: for every pair $a, b \in T_\alpha$, with $a < b$, there exist $x_1 < \cdots < x_m$ in T_α, integers $n_{x_1}, \ldots, n_{x_m}$ and $k < m$ such that $g^{-1}(a) = (b(x_i, n_{x_i}))_1^k$ and $g^{-1}(b) = (b(x_i, n_{x_i}))_1^m$.*

This sounds very complicated but all it is saying is that if you have a tree on a Banach space X, isomorphic to a replacement tree $T(\alpha, \beta)$, then you can replace each β-subtree by a normalized vector in the linear span of a node of that tree, and refine to get a block tree of order α.

Proof: As usual we prove this by induction on α for an arbitrary β. The result is obvious for $\alpha = 1$ and the only non-obvious case is the successor case.

Assume that the lemma is true for α. Let T be a tree on X of order $\beta\cdot(\alpha+1)$ isomorphic to $T(\alpha+1,\beta)$, let $F\colon T\to T_{\alpha+1}$ be the map with $F^{-1}(x)=\bigcup_I T_n(x)$ where $T_n(x)\simeq T_\beta$, and let $b(x,n)$ be given for each $x\in T_{\alpha+1}$, $n\in I$.

By construction of the replacement trees, $T^{\beta\cdot\alpha}\simeq T_\beta$, and in fact $T^{\beta\cdot\alpha}=F^{-1}(\{\gamma+1\})=T_1(\{\gamma+1\})$, where $T_{\alpha+1}=\{a\cup\{\gamma+1\}\colon a\in T_\alpha\}\cup\{\{\gamma+1\}\}$ from Definition 3.1. After each terminal node of $T^{\beta\cdot\alpha}$ lies a tree isomorphic to $T(\alpha,\beta)$. Let these trees be $(t_j)_1^\infty$. Let $j_0\ge 1$ be such that $t_{j_0}>b(\{\gamma+1\},1)$; then $t_{j_0}\simeq T(\alpha,\beta)$ and so the lemma applies giving us a block tree $t'_{j_0}\preceq t_{j_0}$ and $g\colon t'_{j_0}\overset{\sim}{\to}T_\alpha$ as in the statement. Now let

$$T'=\{(b(\{\gamma+1\},1),u_1,\ldots,u_m)\colon (u_i)_1^m\in t'_{j_0}\}\cup\{(b(\{\gamma+1\},1))\}$$

and let $G\colon T'\overset{\sim}{\to}T_{\alpha+1}$ by

$$G(a)=\begin{cases} g((u_i)_1^m)\cup\{\gamma+1\}, & a=(b(\{\gamma+1\},1),u_1,\ldots,u_m),\\ \{\gamma+1\}, & a=(b(\{\gamma+1\},1));\end{cases}$$

then G,T' clearly satisfy the lemma.

The proof where α is a limit ordinal just involves taking the union of the previous trees and functions. ∎

Proof of Theorem 1.1 for $p=1$: Let T be an ℓ_1-K-tree of order α^2 on X. We show that there exists $T'\preceq T$ such that T' is an ℓ_1-$\sqrt{K}$-tree of order α.

By Lemmas 3.3 and 3.7, $T(\alpha,\alpha)$ is isomorphic to a subtree of T and so we may assume that in fact $T(\alpha,\alpha)$ is isomorphic to T. Now let $F\colon T\to T_\alpha$ be the map from Definition 3.5 with $F^{-1}(x)=\bigcup_I T_n(x)$, $T_n(x)\simeq T_\alpha$ for every $x\in T_\alpha$ and $n\in I$.

For each $x\in T_\alpha$ and $n\in I$ we consider $R(T_n(x))$, the restriction being with respect to T. Note that $R(T_n(x))$ is an ℓ_1-K-tree isomorphic to $T_n(x)$. If there exist $x\in T_\alpha$, $n\in I$ such that $R(T_n(x))$ is an ℓ_1-$\sqrt{K}$-tree we are finished, since $R(T_n(x))$ has order α. Otherwise let $(z_1,\ldots,z_m)$ be a node of $R(T_n(x))$ which is not $\sqrt{K}$ equivalent to the unit vector basis of ℓ_1^m and let $b(x,n)=\sum_1^m a_iz_i$ where $(a_i)_1^m\subset\mathbf{R}$, $\sum_1^m|a_i|>\sqrt{K}$ and $\|b(x,n)\|=1$. Thus $b(x,n)$ is a block of $R(T_n(x))$.

By Lemma 4.2 there exists $T'\preceq T$ of order α whose nodes are $(b(x_i,n_{x_i}))_1^m$ for some n_{x_i} where $\{x_1<\cdots<x_m\}=\{x'\in T\colon x'\le x\}$ for each $x\in T_\alpha$. We need only show that this tree has constant $\sqrt{K}$. Let $(y_i)_1^n$ be a node in T' with parent node $z=(z_1,\ldots,z_m)\in T$. Thus there exist subsets $E_i\subset\{1,\ldots,m\}$, $E_1<\cdots<E_n$ (where $E<F$ means $\max E<\min F$) such that $y_i=\sum_{k\in E_i}a_kz_k$

for each i and satisfying:

$$1 = \|y_i\| = \Big\|\sum_{E_i} a_k z_k\Big\| < \frac{1}{\sqrt{K}} \sum_{E_i} |a_k|.$$

Let $(b_i)_1^n \subset \mathbf{R}$; then

$$\begin{aligned}\Big\|\sum_{i=1}^n b_i y_i\Big\| &= \Big\|\sum_{i=1}^n b_i \sum_{k\in E_i} a_k z_k\Big\| \\ &\geq \frac{1}{K}\sum_1^n |b_i| \sum_{k\in E_i} |a_k| \\ &> \frac{1}{K}\sum_1^n |b_i| \cdot \sqrt{K} \\ &= \frac{1}{\sqrt{K}}\sum_1^n |b_i|\end{aligned}$$

as required. These last few lines are James' argument.

Now, if we choose the smallest n so that $K^{1/2^n} \leq 1+\varepsilon$, then we can iterate this argument to show that if T is an ℓ_1-K-tree of order α^{2^n}, then there exists $T' \preceq T$ such that T' is an ℓ_1-$(1+\varepsilon)$-tree of order α, which proves the theorem. ∎

Remark 4.3:

(i) The proof of Theorem 1.1 for $p = \infty$ is very similar to that for $p = 1$, except that given an ℓ_∞-K-tree T on X of order α^{2^n} for n sufficiently large, we choose a block tree $T' \preceq T$ of order α to obtain $\|\sum_1^n b_i y_i\| \leq (1+\varepsilon)\sup_i |b_i|$ for all nodes $(y_i)_1^n \in T'$, and then the lower estimate follows automatically according to James [J].

(ii) The proof of the theorem also gives some fixed points—that is, ordinals α such that if we have an ℓ_1-K-tree of order α, then for any $\varepsilon > 0$ we can get a block tree of this which is an ℓ_1-$(1+\varepsilon)$-tree also of order α. In fact we see from the proof that this is true for every countable ordinal α which satisfies $\beta < \alpha$ implies $\beta^n < \alpha$ for each $n \geq 1$. From basic results on ordinals we see that α satisfies this condition if and only if α is of the form $\alpha = \omega^{\omega^\gamma}$ for some ordinal γ (see Fact 5.3 below).

5. Calculating the ℓ_1-index of a Banach space

Definition 5.1: A **block basis tree** on a Banach space X, with respect to a basis $(e_i)_1^\infty$ for X, is a tree T on X such that every node $(x_i)_1^n$ of T is a block basis of $(e_i)_1^\infty$. Moreover, if T is also an ℓ_1-K-tree, then we say T is an **ℓ_1-K-block basis tree**.

Definition 5.2: Let X be a separable Banach space and for each $K \geq 1$ set

$$I(X,K) = \sup\{o(T)\colon T \text{ is an } \ell_1\text{-}K\text{-tree on } X\}.$$

Bourgain's ℓ_1-index of X [B] is then given by

$$I(X) = \sup_{1 \leq K < \infty} \{I(X,K)\}.$$

By Bourgain, $I(X) < \omega_1$ if and only if X does not contain ℓ_1.

The block basis index is the analogous index to $I(X)$ except that it is only defined on block basis trees. For a Banach space X with a basis (e_i), and $K \geq 1$, set

$$I_b(X,K,(e_i)) = \sup\{o(T)\colon T \text{ is an } \ell_1\text{-}K\text{-block basis tree w.r.t. } (e_i) \text{ on } X\}.$$

The **block basis index** is then given by

$$I_b(X,(e_i)) = \sup\{I_b(X,K,(e_i))\colon 1 \leq K < \infty\}.$$

When the basis in question is fixed we shall write $I_b(X,K)$ rather than $I_b(X,K,(e_i))$ etc. It is worth recalling here that $I_b(X)$ is not in general independent of the basis. It is clear, however, that $I_b(X,K,(e_i)) \leq I(X,K)$ for every X, K and (e_i).

We next state some facts about ordinals. The proofs may be found in Monk [M].

FACT 5.3: *Let α be an infinite countable ordinal. Then the following statements hold:*

(i) *There exist $k \geq 1$, (countable) ordinals $\theta_1 > \cdots > \theta_k \geq 0$ and $n_i \geq 1$ ($i = 1,\ldots,k$), uniquely determined by α, such that*

$$\alpha = \omega^{\theta_1} \cdot n_1 + \cdots + \omega^{\theta_k} \cdot n_k.$$

This is the **Cantor normal form** *of an ordinal.*

(ii) *For all $\beta < \alpha$, $\beta \cdot 2 < \alpha$ if and only if there exists $\gamma < \omega_1$ such that $\alpha = \omega^\gamma$.*

(iii) *For all $\beta < \alpha$, $\beta^2 < \alpha$ if and only if there exists $\gamma < \omega_1$ such that $\alpha = \omega^{\omega^\gamma}$.*

(iv) *If $\alpha = \omega^{\theta_1} \cdot n_1 + \cdots + \omega^{\theta_k} \cdot n_k$, then $\omega \cdot \alpha = \alpha$ if and only if $\theta_k \geq \omega$.*

(v) *If $\alpha = \omega^{\theta_1} \cdot n_1 + \cdots + \omega^{\theta_k} \cdot n_k$, then $\alpha \cdot \omega = \omega^{\theta_1 + 1}$.*

Our first result of this section is to show how we may refine ℓ_1-trees in a Banach space with a basis to get ℓ_1-block basis trees, and explain how this relates to the indices. We then show that both $I(X)$ and $I_b(X)$ are of the form ω^α for some α, and that if $\alpha \geq \omega$ for either index, then the indices are the same. The block basis trees are much easier to work with, and once we have the block basis index of a space we have a good idea what the index is. In the second part of this section we use this idea to calculate the index of some Tsirelson type spaces.

NOTATION. For a Banach space X let $B(X) = \{x \in X\colon \|x\| \leq 1\}$ and $S(X) = \{x \in X\colon \|x\| = 1\}$ denote the unit ball and unit sphere of X respectively. If $(x_i)_{i\in I} \subset X$, where $I \subset \mathbf{N}$, let $[x_i]_{i\in I}$ be the closed linear span of these vectors.

If X is a Banach space with basis $(e_i)_1^\infty$ let $E_n = [e_i]_1^n$, let $P_n\colon X \to E_n$ be the basis projection onto E_n given by $P_n(\sum a_i e_i) = \sum_1^n a_i e_i$, and let $X_n = [e_i]_{n+1}^\infty$. Finally, we define the support of $x \in X$ with respect to $(e_i)_1^\infty$ as $\operatorname{supp}(x) = \{n \geq 1\colon (P_n - P_{n-1})(x) \neq 0\}$. Thus, if $x = \sum_F a_i e_i$ with $a_i \neq 0$ for $i \in F$, then $\operatorname{supp}(x) = F$. If $x = (x_1, \ldots, x_n)$ is a sequence of vectors, then $\operatorname{supp}(x) = \bigcup_1^n \operatorname{supp}(x_i)$. In the following X will always denote a separable Banach space not containing ℓ_1.

PROPOSITION 5.4: *Let X have a basis; then $I(X,K) \geq \omega \cdot \alpha$ implies that $I_b(X, K+\varepsilon) \geq \alpha$ for every $\varepsilon > 0$.*

We first prove the following elementary lemma:

LEMMA 5.5: *Let X be a Banach space with basis $(e_i)_1^\infty$ and let T be an ℓ_1-tree of order ω on X; then for each $n \geq 1$ there exists a block x of T with $P_n x = 0$.*

Proof: There exists $m > n$ such that the linear space spanned by $(y_i)_1^m \in T$ has dimension greater than n. Thus the restriction of P_n to $[y_i]_1^m$ is not one to one and hence there exists $x \in [y_i]_1^m$ with $\|x\| = 1$ and $P_n x = 0$. ∎

Proof of Proposition 5.4: If $I(X,K) \geq \omega \cdot \alpha$, then there exists an ℓ_1-K-tree T on X of order $\omega \cdot \alpha$ and this in turn, by Lemma 3.7, has an ℓ_1-K-subtree T' isomorphic to $T(\alpha, \omega)$. Thus we may assume that T itself is isomorphic to $T(\alpha, \omega)$. We prove the following statement:

> For all $\alpha < \omega_1$, each $l \geq 0$, and every $\varepsilon > 0$, if T is an ℓ_1-K-tree isomorphic to $T(\alpha, \omega)$, then there exists an ℓ_1-K-block tree T' of T of order α such that for any node $(y_i)_1^m \in T'$ there exists $l = k(1) < \cdots < k(m+1)$ with $\|y_i - P_{k(i+1)} y_i\| < \varepsilon$ and $P_{k(i)} y_i = 0$ $(i = 1, \ldots, m)$.

We induct on α; the statement is clear for $\alpha = 1$ by Lemma 5.5. Suppose we have proved the statement for α, and let $T \simeq T(\alpha+1,\omega)$. Let $F\colon T \to T_{\alpha+1}$ be the map $F^{-1}(x) = \bigcup_I T_n(x)$ ($I = \{1\}$ or $\mathbf{N}$), from Definition 3.5, such that $T_n(x) \simeq T_\omega$ for each n and x, and the $T_n(x)$'s are mutually incomparable. Let z be the unique initial node of $T_{\alpha+1}$. By Lemma 5.5 we can find a block $b(1,z)$ of $T_1(z)$ such that $P_l b(1,z) = 0$ and we can find $l' > l$ such that $\|b(1,z) - P_{l'} b(1,z)\| < \varepsilon$. Let $\tilde{T}$ be a subtree of T isomorphic to $T(\alpha,\omega)$ with $b(1,z) < \tilde{T}$. Applying the induction hypothesis to $R(\tilde{T})$ we obtain $\tilde{T}' \preceq R(\tilde{T})$ such that $P_{l'} y_i = 0$ for every node $(y_i)_1^m \in \tilde{T}'$. Let $T' = \{(b(1,z), y_1, \ldots, y_m)\colon (y_i)_1^m \in \tilde{T}'\} \cup \{(b(1,z))\}$. Then T' is the required block tree.

Now let α be a limit ordinal and suppose we have proved the statement for each $\beta < \alpha$. Let (α_n) be the sequence of ordinals increasing to α such that $T = \bigcup_1^\infty T(n)$ where the trees $T(n)$ are mutually incomparable and $T(n) \simeq T(\alpha_n,\omega)$. Applying the hypothesis to each $T(n)$ we obtain block trees $T(n)' \preceq T(n)$. Then $T' = \bigcup_1^\infty T(n)'$ is the required block tree.

Thus, if we have an ℓ_1-K-tree T of order $\omega \cdot \alpha$ and $\varepsilon' > 0$, then let T' be the ℓ_1-K-block tree of T from above. For each node $(y_i)_1^m$ of T' let $(k(i))_1^{m+1} \subset \mathbf{N}$ be the sequence from above and let

$$v_i = \frac{P_{k(i+1)} y_i}{\|P_{k(i+1)} y_i\|} \quad (i = 1, \ldots, m).$$

The sequence $(v_i)_1^m$ is a uniform perturbation of a basis K equivalent to the unit vector basis of ℓ_1^m. Hence, if ε' is chosen sufficiently small, then $(v_i)_1^m$ is $K+\varepsilon$ equivalent to the unit vector basis of ℓ_1^m. This completes the proof since if we replace the nodes $(y_i)_1^m$ with $(v_i)_1^m$ as above, then we obtain $\tilde{T}$, a block basis tree of order α and constant $(K+\varepsilon)$, so that $I_b(X, K+\varepsilon) \geq \alpha$ as required. ∎

THEOREM 5.6: *Let X be a Banach space with a basis; then $I_b(X) = \omega^\alpha$ for some $\alpha < \omega_1$.*

To prove this theorem we need some preliminary results. We first show that there is no ℓ_1-K-block basis tree whose order is the same as the block basis index, and hence $I_b(X)$ must be a limit ordinal. Then we show that $\beta < I_b(X)$ implies that $\beta \cdot 2 < I_b(X)$, which completes the proof.

LEMMA 5.7: *Let X be a Banach space with a basis and $K \geq 1$; then $I_b(X,K) \neq I_b(X)$. In particular $I_b(X)$ is a limit ordinal.*

Proof: We prove by induction on α that for every Banach space X with a basis and any $K \geq 1$, if $I_b(X,K) = \alpha$, then $I_b(X) > \alpha$. This is trivial for $\alpha = 1$.

Let the result be true for α and suppose, if possible, that it is false for $\alpha+1$. Let X be a Banach space with basis $(e_i)_1^\infty$ and $K \geq 1$ such that $I_b(X,K) = I_b(X) = \alpha+1$. Now there exists an ℓ_1-K-block basis tree T of order $\alpha+1$ isomorphic to the minimal tree $T_{\alpha+1}$. Let $x = (x_1)$ be the unique initial node of T, let $k = \max(\operatorname{supp} x_1)$, let X_k be the subspace of X spanned by $(e_i)_{i>k}$ and let $T(\alpha) = \{(y_i)_1^m\colon y = (x_1, y_1, \ldots, y_m) \in T \text{ and } y > x\}$. Clearly $T(\alpha)$ is an ℓ_1-K-block basis tree on X_k of order α, and so $I_b(X_k) > \alpha$, otherwise $I_b(X_k, K) = \alpha = I_b(X_k)$ contradicting our assumption. Thus there exists an ℓ_1-block basis tree T' on X_k of order $\alpha+1$ for some constant $K' \geq 1$. But now the tree $\tilde{T} = \{(x_1, u_1, \ldots, u_l)\colon (u_1, \ldots, u_l) \in T'\} \cup \{(x_1)\}$ is an ℓ_1-block basis tree on X of order $\alpha+2$ for some constant K'' contradicting the assumption that $I_b(X) = \alpha+1$. This proves the result for $\alpha+1$.

Let α be a limit ordinal and suppose the result is true for every $\alpha' < \alpha$, but false for α. Again let X be a Banach space with basis $(e_i)_1^\infty$, $K \geq 1$ such that $I_b(X,K) = I_b(X) = \alpha$ and T an ℓ_1-K-block basis tree of order α isomorphic to the minimal tree T_α. By Lemma 3.2 there exists a sequence of ordinals (α_n) such that $\alpha = \sup_n(\alpha_n + 1) = \sup_n \alpha_n$ and mutually incomparable trees t_n for each n such that $t_n \simeq T_{\alpha_n+1}$ and $T = \bigcup_n t_n$. For each n let $z_n = (w_i^n)_1^{k_n}$ be the unique initial node of t_n and let $t'_n = \{(y_i)_1^m\colon y = (w_1^n, \ldots, w_{k_n}^n, y_1, \ldots, y_m) \in t_n \text{ and } y > z_n\}$, a tree isomorphic to T_{α_n}. Clearly $T' = \bigcup_n t'_n$ is a tree on X_1 with order α. Let $\tilde{T} = \{(e_1, u_1, \ldots, u_l)\colon (u_1, \ldots, u_l) \in T'\} \cup \{(e_1)\}$. This is an ℓ_1-block basis tree of order $\alpha+1$, contradicting the assumption that $I_b(X) = \alpha$. This proves the first part of the lemma.

Suppose, if possible, that $I_b(X) = \alpha+1$ for some α. Then there exists an ℓ_1-K-block basis tree T of order $\alpha+1$ for some K contradicting the previous result. ∎

LEMMA 5.8: *Let X be a Banach space with basis $(e_i)_1^\infty$. If $\beta < I_b(X)$, then there exists $K > 1$ such that $I_b(X_n, K) \geq \beta$ for every $n \geq 1$.*

Proof: The result is trivial for $\beta < \omega$. Suppose first that $\beta < I_b(X)$ is a limit ordinal and let T be an ℓ_1-K-block basis tree on X of order β. Let $T(n) = \{(x_i)_{n+1}^l\colon \exists (x_i)_1^l \in T \text{ with } l > n\}$. $T(n)$ is clearly a block tree of T and an ℓ_1-K-block basis tree on X_n. Moreover, $o(T(n)) = \beta$, otherwise $o(T) \leq o(T(n)) + n < \beta$, a contradiction.

Now let $\beta < I_b(X)$ be a successor ordinal greater than ω; then $\beta = \beta' + k$ for some limit ordinal $\beta' \geq \omega$ and $k \geq 1$. From the limit ordinal case there exists $K > 1$ such that $I_b(X_m, K) \geq \beta'$ for every $m \geq 1$. Now, X contains ℓ_1^n's uniformly so there exists $m > n$ and a normalized block basis $(x_i)_1^k$ of $[e_i]_{n+1}^m$ which is K equivalent to the unit vector basis of ℓ_1^k. Let T be an ℓ_1-K-block basis tree on X_m of order β' and let $T(n) = \{(x_1, \dots, x_k, u_1, \dots, u_l)\colon (u_1, \dots, u_l) \in T\} \cup \{(x_1, \dots, x_k), \dots, (x_1)\}$. Then $T(n)$ is an ℓ_1-block basis tree on X_n of order $\beta' + k = \beta$ and some constant which depends only on K. ∎

Proof of Theorem 5.6: We show that if $\beta < I_b(X)$, then $\beta \cdot 2 < I_b(X)$, which is enough to prove the theorem by Fact 5.3 (ii). Let $\beta < I_b(X)$ and let T be an ℓ_1-K-block basis tree on X of order β. For each n let $T(n)$ be an ℓ_1-K-block basis tree on X_n of order β from Lemma 5.8. Let (a_i) be the collection of terminal nodes of T and for each $i \geq 1$ let $n(i) = \max(\operatorname{supp} a_i)$. Finally, setting $\tilde{T}(n(i)) = \{a_i \cup x\colon x \in T(n(i))\}$, we have that $\tilde{T} = T \cup (\bigcup_i \tilde{T}(n(i)))$ is an ℓ_1-block basis tree of order $\beta \cdot 2$ and hence $I_b(X) > \beta \cdot 2$ as required. ∎

THEOREM 5.9: *Let X be a separable Banach space; then $I(X) = \omega^\alpha$ for some $\alpha < \omega_1$.*

The proof of this theorem is similar to that of Theorem 5.6, but without a basis for X we have to work harder.

LEMMA 5.10: *Let T be a countable tree of order $\alpha < \omega_1$, M the collection of maximal nodes of T, $M = \bigcup_{i=1}^n M_i$ a partition of M, and*

$$T_i = \{x \in T\colon x \leq m \text{ for some } m \in M_i\}.$$

Then $o(T_i) = \alpha$ for some $1 \leq i \leq n$.

Proof: We prove by induction on α. The result is obvious for $\alpha = 1$. Suppose it is true for α, and let T be a countable tree of order $\alpha + 1$ with M, M_i, T_i as above. Let (a_j) be the sequence of initial nodes of T and $t_j = \{x \in T\colon x \geq a_j\}$. Clearly the t_j's are mutually incomparable and $T = \bigcup_j t_j$, hence $o(t_{j_0}) = \alpha + 1$ for some j_0. Let $t' = \{x \in T\colon x > a_{j_0}\}$, then $o(t') = \alpha$. Now, $M = \bigcup_{i=1}^n M_i$ also partitions the terminal nodes of t' and setting

$$t'_i = \{x \in t'\colon x \leq m \text{ for some } m \in M_i\}$$

we have $o(t'_{i_0}) = \alpha$ for some i_0 by assumption. Now $\{a_{j_0}\} \cup t'_{i_0}$ is a tree of order $\alpha + 1$ and $\{a_{j_0}\} \cup t'_{i_0} \subseteq T_{i_0}$. Thus $o(T_{i_0}) = \alpha + 1$ as required.

Let α be a limit ordinal and suppose the result is true for each $\alpha' < \alpha$. Write $T = \bigcup t_k$ as a union of mutually incomparable trees t_k of order α_k where $\sup_k \alpha_k = \alpha$. Given M, M_i, T_i as above let

$$t_{k,i} = \{x \in t_k\colon x \leq m \text{ for some } m \in M_i\}$$

and let $i(k) \in \{1, \ldots, n\}$ satisfy $o(t_{k,i(k)}) = \alpha_k$ for each k, by assumption. Let $N_i = \{k \geq 1: i(k) = i\}$, then N_i must be infinite for some i_0, so let $N_{i_0} = (k_j)_1^\infty$. Now for each j we have $t_{k_j,i(k_j)} = t_{k_j,i_0} \subseteq T_{i_0}$ and the trees t_{k_j,i_0} are mutually incomparable, thus $o(\bigcup_j t_{k_j,i_0}) = \alpha$ which implies $o(T_{i_0}) = \alpha$ as required. ∎

LEMMA 5.11: *Let X be a separable Banach space not containing ℓ_1 and $K \geq 1$; then $I(X, K) \neq I(X)$. In particular $I(X)$ is a limit ordinal.*

Proof: Let $I(X, K) = \alpha$ for some $\alpha < \omega_1$ and let T be an ℓ_1-K-tree on X of order α. Recall that a Banach space X is $\mathcal{L}_1$-K if there exists a collection $(E_n)_1^\infty$ of finite dimensional subspaces of X with $d(E_n, \ell_1^{\dim E_n}) \leq K$ for every n, and for each finite set $F \subset X$ and all $\varepsilon > 0$ there exists n such that the distance from x to E_n is less than ε for all x in F. Also recall that every infinite dimensional $\mathcal{L}_1$ space contains ℓ_1. See [LT] for more information on $\mathcal{L}_1$ spaces.

Now let M be the set of maximal nodes of T. Clearly this defines a collection of finite dimensional subspaces $[x_i]_1^n$ such that $d([x_i]_1^n, \ell_1^n) \leq K$, where $(x_i)_1^n \in M$. Thus, since X doesn't contain ℓ_1, it is not a $\mathcal{L}_1$ space and hence there exist $F = \{z_1, \ldots, z_r\} \subseteq S(X)$ and $\varepsilon > 0$ such that for each $m = (x_i)_1^n \in M$ there exists $i(m) \in \{1, \ldots, r\}$ with $d(z_{i(m)}, S([x_i]_1^n)) > \varepsilon$. For $i = 1, \ldots, r$ set $M_i = \{m \in M: i(m) = i\}$. Then $M = \bigcup_1^r M_i$ partitions M and defines $T = \bigcup_1^r T_i$ as in Lemma 5.10. So, from the lemma, we have $o(T_{i_0}) = \alpha$ for some $i_0 \leq r$. Let $T' = \{(z_{i_0}, u_1, \ldots, u_m): (u_1, \ldots, u_m) \in T_{i_0}\} \cup \{(z_{i_0})\}$; then this is an ℓ_1-tree on X, for some constant $K' = K'(K, \varepsilon)$, of order $\alpha + 1$. Thus $I(X) > \alpha = I(X, K)$ which completes the first part of the proof. The argument that $I(X)$ is a limit ordinal is the same as for $I_b(X)$. ∎

LEMMA 5.12: *Let T be a tree on X of order α, where α is a limit ordinal. Let $F \subset S(X^*)$ be finite and $X_F = \{x \in X: x^*(x) = 0\ \forall x^* \in F\}$. Then there exists a block tree T' of T with $o(T') = \alpha$ and $T' \subseteq X_F$.*

Proof: Let $|F| = n$. We note that α is a limit ordinal if and only if $\alpha = \omega \cdot \beta$ for some ordinal β, and prove the lemma by induction on β.

For $\beta = 1$, $\alpha = \omega$, and let T be isomorphic to T_ω. Notice that if $(x_i)_1^{n+1} \in T$, then there exists $x \in S([x_i]_1^{n+1})$ with $x \in X_F$. Thus for each k there exists a node $(x_i^k)_{i=1}^l \in T$, for l sufficiently large, from which we may extract a normalized block basis $(y_j^k)_{j=1}^k$ of $(x_i^k)_{i=1}^l$ which is contained in X_F and such that $T' = \{(y_1^k, \ldots, y_j^k): 1 \leq j \leq k,\ k \geq 1\}$ is a block tree of T. This is now the required tree.

Suppose the result is true for β and let $\alpha = \omega \cdot (\beta + 1) = \omega \cdot \beta + \omega$ and T be a tree of order α. Since T has a subtree isomorphic to T_α we may assume $T \simeq T_\alpha$.

Now $T^{\omega\cdot\beta}$ is isomorphic to T_ω and we apply the case $\beta = 1$ to obtain a block tree $\tilde{T} \lesssim T^{\omega\cdot\beta}$ of order ω, contained in X_F. Let $(\tilde{a}_i)$ be the sequence of terminal nodes in $\tilde{T}$ and a_i the parent node of $\tilde{a}_i$ in $T^{\omega\cdot\beta}$ for each i. Let $T(i) = \{x \in T\colon x > a_i\}$, then $o(T(i)) \geq \omega\cdot\beta$. Thus we may apply the induction hypothesis to $R(T(i))$ (the restricted tree from Definition 4.1) for each i to obtain block trees $T(i)' \lesssim T(i)$ with $o(T(i)') = \omega\cdot\beta$ and $T(i)' \subset X_F$. Finally, $T' = \tilde{T} \cup (\bigcup_i T(i)')$ is the required tree of order α.

Let β be a limit ordinal and suppose the result is true for all $\beta' < \beta$. Let (β_n) be the increasing sequence of ordinals whose limit is β, then $\alpha = \omega\cdot\beta = \sup_n \omega\cdot\beta_n$ so that if T is a tree of order α and then T contains mutually incomparable trees of order $\omega\cdot\beta_n$ for each β_n. We apply the hypothesis to each of these trees to obtain the result. ∎

Proof of Theorem 5.9: By Lemma 5.11, $I(X) = \alpha$ for some limit ordinal α. We show that if $\beta < \alpha$ is a limit ordinal, then $\beta\cdot 2 < \alpha$. It follows that if $\beta < \alpha$ is a successor ordinal, then $\beta\cdot 2 < \alpha$. This is enough to prove the theorem by Fact 5.3.

Let T be an ℓ_1-K-tree of order β for some K. If $(x_i)_1^n \in T$ let $F = F((x_i)_1^n) \subset S(X^*)$ be a finite set which 1-norms a $(1/2)$-net in $S([x_i]_1^n)$. Choose by Lemma 5.12 $T_{(x_i)_1^n,F}$ a block tree of T of order β contained in X_F. Let (a_k) be the collection of maximal nodes of T and if $a_k = (x_i^k)_1^n$, $F_k = F(a_k)$, let

$$T(k) = \{a_k \cup x\colon x \in T_{a_k,F}\}.$$

Thus $T' = T \cup (\bigcup_k T(k))$ is an ℓ_1-$6K$-tree of order $\beta\cdot 2$ as required, and hence $\beta\cdot 2 < \alpha$. ∎

COROLLARY 5.13: *Let X have a basis. If $I(X) \geq \omega^\omega$, then $I(X) = I_b(X)$.*

Proof: Let $\alpha > \omega$, and suppose $I(X) = \omega^\alpha$. Then for every β with $\omega \leq \beta < \alpha$ there exists K such that $I(X,K) \geq \omega\cdot\omega^\beta = \omega^\beta$. Thus $I_b(X,K') \geq \omega^\beta$ for some K' by Proposition 5.4, and hence $I_b(X) > \omega^\beta$ for every $\beta < \alpha$. If α is a limit ordinal, then $\omega^\alpha = \sup_{\beta<\alpha}\omega^\beta$ and so $I_b(X) \geq \omega^\alpha = I(X)$. Otherwise $\alpha = \alpha' + 1$, where $\alpha' \geq \omega$ and $I_b(X) > \omega^{\alpha'}$, which implies $I_b(X) \geq \omega^\alpha = I(X)$ since $I_b(X) = \omega^\gamma$ for some γ by Theorem 5.6. In either case we know that $I(X) \geq I_b(X)$ and so they are equal.

If $I(X) = \omega^\omega$, then $I(X) > \omega^{n+1}$ for every $n \geq 1$ and hence $I_b(X) > \omega^n$ for every $n \geq 1$ by Proposition 5.4. Thus $I_b(X) \geq \omega^\omega = I(X)$ and so $I(X) = I_b(X)$ as required. ∎

COROLLARY 5.14: *If $I(X) = \omega^n$, then $I_b(X) = \omega^m$ where $m = n$ or $n-1$.*

Proof: This follows from similar arguments to those for the previous corollary. ∎

Remark 5.15: We collect together some notes about which values α may take when $I(X) = \omega^\alpha$.

(i) If X does not contain ℓ_1^n's uniformly, then $I(X) = \omega = I_b(X)$. Also, if X contains ℓ_1^n's uniformly, then $I(X) \geq \omega^2$. It is easy to see that $I_b(c_0) = \omega$ (where the block basis index is calculated with respect to the unit vector basis for c_0) and so $I(c_0) = \omega^2$ by Corollary 5.14. Thus the two ordinal indices may indeed differ. In fact, by Remark 5.21 below, for each $n \geq 1$ there exists a Banach space X_n with $I_b(X_n) = I(X_n) = \omega^{n+1}$ and for each $n \geq 1$ there exists a Banach space Y_n with $I_b(Y_n) = \omega^n$ while $I(Y_n) = \omega^{n+1}$.

(ii) If $I(X) < \omega^\omega$, then it is possible for a space X to have two bases (x_i) and (y_i) with $I_b(X,(x_i)) \neq I_b(X,(y_i))$. Indeed, for each $n \geq 1$ let H_n be the span of the first 2^n Haar functions in $C(\Delta)$ (where Δ is the Cantor set on $[0,1]$); if $X = (\sum H_n)_{c_0}$, then $X \simeq c_0$. Thus, if (x_i) is a basis for X equivalent to the unit vector basis of c_0, then $I_b(X,(x_i)) = \omega$. However, if (y_i) is the basis for X consisting of the Haar bases for the H_n's strung together, then, since each basis for H_n admits a block basis of length n which is 1-equivalent to the unit vector basis of ℓ_1^n, we obtain $I_b(X,(y_i)) = \omega^2$. By Corollary 5.14 the block basis indices for different bases can only differ by a factor of ω.

(iii) We note here that there are some ordinals α for which there are no spaces X with index $I(X) = \omega^\alpha$. In particular, if α is a limit ordinal, then there is no space X with $I(X) = \omega^{\omega^\alpha}$. Otherwise, let $I(X) = \omega^{\omega^\alpha}$; then for all $\alpha' < \alpha$ there is some K such that there exists an ℓ_1-K-tree of order $\omega^{\omega^{\alpha'}}$, which we may then refine to get an ℓ_1-$(1+\varepsilon)$-block tree of order $\omega^{\omega^{\alpha'}}$ for any $\varepsilon > 0$, by Remark 4.3 (ii). Hence X contains a block basis tree of constant 2 and order ω^{ω^α} (taking the union of these trees) and so $I(X) \geq \omega^{\omega^\alpha+1}$.

(iv) By Remark 5.21 below, for every $\alpha < \omega_1$ there exists a Banach space X with $I(X) = \omega^{\alpha+1}$, and by Theorem 5.19 below, there exists a Banach space $Y = T(S_{\omega^\alpha}, 1/2)$ with $I(Y) = \omega^{\omega^{\alpha+1}}$.

(v) If X is asymptotic ℓ_1 (see for example [OTW] for the definition of this), then $I_b(X) \geq \omega^\omega$ and so $I(X) = I_b(X)$.

QUESTION 1: *For which limit ordinals α do there exist Banach spaces X with index $I(X) = \omega^\alpha$?*

We have already shown that there exist Banach spaces with index $\omega^{\alpha+1}$ for every $\alpha < \omega_1$. We have also shown that we cannot have indices of the form ω^{ω^α} for α a limit ordinal, and that we do have spaces with index of the form $\omega^{\omega^{\alpha+1}}$, but this leaves the question open for all other limit ordinals.

This completes the first part of the section. We now apply some of these results and methods to calculating the ℓ_1 index of some Tsirelson spaces.

Definition 5.16: Let $E, F \subseteq \mathbf{N}, n \geq 1$. We write $E < F$ if $\max E < \min F$ and $n < E$ if $\{n\} < E$. Let $\mathcal{M}, \mathcal{N}$ be collections of finite sets of integers and $K = (k_i)_1^\infty \subseteq \mathbf{N}$. We define

$$\mathcal{M}[\mathcal{N}] = \{\bigcup_1^k F_i \colon F_i \in \mathcal{N} \ (i = 1, \ldots, k) \text{ and } \exists E = \{m_1, \ldots, m_k\} \in \mathcal{M}$$
$$\text{with } m_1 \leq F_1 < m_2 \leq F_2 < \cdots < m_k \leq F_k;\ k \geq 1\}$$

and $\mathcal{M}(K) = \{\{k_i \colon i \in E\} \colon E \in \mathcal{M}\}$.

The **Schreier sets,** $\mathcal{S}_\alpha$ [AA], for each $\alpha < \omega_1$ are defined inductively as follows: Let $\mathcal{S}_0 = \{\{n\} \colon n \geq 1\} \cup \{\emptyset\}$, $\mathcal{S}_1 = \{F \subset \mathbf{N} \colon |F| \leq F\} = \mathcal{S}_1[\mathcal{S}_0]$. If $\mathcal{S}_\alpha$ has been defined let $\mathcal{S}_{\alpha+1} = \mathcal{S}_1[\mathcal{S}_\alpha]$. If α is a limit ordinal with $\mathcal{S}_{\alpha'}$ defined for each $\alpha' < \alpha$ choose an increasing sequence of ordinals (α_n) with $\alpha = \sup_n \alpha_n$ and let $\mathcal{S}_\alpha = \bigcup_{n=1}^\infty \{F \in \mathcal{S}_{\alpha_n} \colon n \leq F\}$.

For $n \geq 1$ let $(\mathcal{S}_\alpha)^n = \{F = \bigcup_1^n F_i \colon F_i \in \mathcal{S}_\alpha,\ F_1 < \cdots < F_n\}$ and let $[\mathcal{S}_\alpha]^n = \mathcal{S}_\alpha[\cdots[\mathcal{S}_\alpha]]$ (n times). A sequence $(E_i)_1^n$ of finite subsets of integers is $\mathcal{S}_\alpha$ admissible if $E_1 < \cdots < E_n$ and $(\min E_i)_1^n \in \mathcal{S}_\alpha$.

Note that $(\mathcal{S}_\alpha, \subseteq)$ forms a tree, Tree$(\mathcal{S}_\alpha)$, of order ω^α and $([\mathcal{S}_\alpha]^n, \subseteq)$ forms a tree, Tree$([\mathcal{S}_\alpha]^n)$, of order $\omega^{\alpha \cdot n}$ (see e.g. [AA]).

Definition 5.17: We first define c_{00} to be the linear space of all real sequences with finite support, and let $(e_i)_1^\infty$ be the unit vector basis of c_{00}. If $E \subset \mathbf{N}$, then let $Ex = \sum_{i \in E} a_i e_i$.

Using the Schreier sets, Argyros [A] defined the **Tsirelson spaces,** $T(\mathcal{S}_\alpha, 1/2)$, for $\alpha < \omega_1$. He showed there exists a norm $\|\cdot\|$ on c_{00} satisfying the implicit equation

$$\|x\| = \max\left(\|x\|_{c_0}, \frac{1}{2}\sup\Big\{\sum_{i=1}^{n}\|E_i x\| \colon (E_i)_1^n \text{ is } \mathcal{S}_\alpha \text{ admissible and } n \geq 1\Big\}\right).$$

The space $T(\mathcal{S}_\alpha, 1/2)$ is the completion of $(c_{00}, \|\cdot\|)$. The standard Tsirelson space T (the dual of Tsirelson's original space [T]) is just $T(\mathcal{S}_1, 1/2)$ [FJ].

Definition 5.18: The **Schreier spaces X_α for** $\alpha < \omega$ are generalizations of Schreier's example [Sch], first discussed in [AA] and [AO]. They are defined in a similar way to Tsirelson space; for each $\alpha < \omega_1$ we define a norm on c_{00} by

$$\Big\|\sum a_i e_i\Big\|_\alpha = \sup_{E \in \mathcal{S}_\alpha}\Big|\sum_{i\in E} a_i\Big|,$$

and then the Schreier space X_α is the completion of $(c_{00}, \|\cdot\|_\alpha)$.

THEOREM 5.19: $I_b(T(\mathcal{S}_\alpha, 1/2)) = \omega^{\alpha\cdot\omega} = I(T(\mathcal{S}_\alpha, 1/2))$.

PROPOSITION 5.20: *For each $\alpha < \omega_1$, for every $\varepsilon > 0$, and for all $m \geq 1$, there exists $n \geq 1$ such that if T is a block basis tree on a Banach space with a basis, and if $\mathcal{F}(T) = \{(\min(\operatorname{supp} x_i))_1^l \colon (x_i)_1^l \in T\}$ satisfies: $\forall F \in \mathcal{F}(T)\ \forall (a_i)_F \subset \mathbf{R}^+$ with $\sum_F a_i = 1$ there exists $G \in (\mathcal{S}_\alpha)^m$ such that $G \subset F$ and $\sum_G a_i \geq \varepsilon$, then $o(T) \leq \omega^\alpha \cdot n$.*

Proof: We prove the result by induction on α. Let $\alpha = 0$, pick $\varepsilon > 0$, $m \geq 1$ and choose n so that $m/n < \varepsilon$. If $o(T) > \omega^0 \cdot n = n$, then there exists $F \in \mathcal{F}(T)$, $|F| > n$. Now, setting $a_i = 1/|F|$ for $i \in F$ gives $\sum_F a_i = 1$ but if $G \in (\mathcal{S}_0)^m$, then $|G| = m$ and $\sum_G a_i = m/|F| < m/n < \varepsilon$, a contradiction.

Suppose the result is true for α. To prove the case $\alpha + 1$ first let $\varepsilon > 0$ be arbitrary and fix $m = 1$. Let $n > 2/\varepsilon$, and let T be a tree with $o(T) \geq \omega^{\alpha+1} \cdot n$. We may assume by Lemma 3.7 that $T \simeq T(n, \omega^{\alpha+1})$ and let $F \colon T \to T_n = \{a_1 < \cdots < a_n\}$ be the map $F^{-1}(a_1) = T_1(a_1)$ and $F^{-1}(a_i) = \bigcup_{n=1}^\infty T_n(a_i)$ $(i > 1)$ where $T_n(a_i) \simeq T_{\omega^{\alpha+1}}$ and the $T_n(a_i)$'s are mutually incomparable. Fix $m_1 = 1$ and $\varepsilon_1 < 1/n$. $T_1(a_1)$ has index $\omega^{\alpha+1} > \omega^\alpha \cdot k\ \forall k$ so $\exists F_1 \in \mathcal{F}(T_1(a_1))$ $\exists (a_i)_{F_1} \subset \mathbf{R}^+$ such that $\sum_{F_1} a_i = 1$ and $\sum_G a_i < \varepsilon$ if $G \in (\mathcal{S}_\alpha)^{m_1}$. Let $(x_i)_1^l \in T_1(a_1)$ be a node such that $F_1 = (\min(\operatorname{supp} x_i))_1^l$. Then there exists i_2 such that $T_{i_2}(a_2) > (x_i)_1^l$.

Choose $m_2 = \max(F_1)$ and $\varepsilon_2 < 1/n$. Repeating the process for the restricted tree $R(T_{i_2}(a_2))$ and m_2, ε_2 up to $R(T_{i_n}(a_n))$ and m_n, ε_n we obtain $F_1 < \cdots < F_n$, $(a_j)_{F_i} \subset \mathbf{R}^+$ such that $\sum_{F_i} a_j = 1$ and $\sum_G a_j < \varepsilon_i$ if $G \subseteq F_i$ and $G \in (\mathcal{S}_\alpha)^{m_i}$. Set $F = \bigcup_1^n F_i$ and $\overline{a}_j = \frac{1}{n} a_j$ for $j \in F$. Let $G \subseteq F$, $G \in \mathcal{S}_{\alpha+1}$. Then $G = \bigcup_1^r G_j$

where $r \leq G_1 < \cdots < G_r$ and $G_j \in \mathcal{S}_\alpha$. Let i be least such that $G \cap F_i \neq \emptyset$, then $r \leq \max(F_i) = m_{i+1} \leq m_l$ $\forall l > i$. Hence if $l > i$, then $G \in (\mathcal{S}_\alpha)^{m_l}$ and so $\sum_{G\cap F_l} \overline{a}_j = \frac{1}{n}\sum_{G\cap F_l} a_j < \varepsilon_l/n$; further, $\sum_{G\cap F_i} \overline{a}_j \leq \sum_{F_i} \overline{a}_j = 1/n$. Thus

$$\sum_G \overline{a}_j < \frac{1}{n}(1+\varepsilon_i+\cdots+\varepsilon_n) < \frac{2}{n} < \varepsilon$$

as we had to show.

For general $m > 1$ we use the same construction, taking $n > 2m/\varepsilon$. Then, for $G \in (\mathcal{S}_{\alpha+1})^m$, each set in $\mathcal{S}_{\alpha+1}$ can contribute at most $(1+\sum \varepsilon_i)/n$ and hence we get the desired contradiction.

Let α be a limit ordinal and suppose the result is true for each $\alpha' < \alpha$. Let (α_i) be the increasing sequence of ordinals, with $\sup_i \alpha_i = \alpha$, which defines $\mathcal{S}_\alpha$. Let $\varepsilon > 0$, $m = 1$, and choose $n > 2/\varepsilon$. Suppose $o(T) \geq \omega^\alpha \cdot n$, and so assume $T \simeq T(n, \omega^\alpha)$; let $F\colon T \to T_n \equiv \{a_1 < \cdots < a_n\}$ be as before, but now with $T_n(a_i) \simeq T_{\omega^\alpha}$. From $\mathcal{F}(T_1(a_1))$ select F_1, $(a_i)_{F_1} \subset \mathbf{R}^+$ arbitrarily. Let $(x_i)_1^l \in T_1(a_1)$ be a node such that $F_1 = (\min(\operatorname{supp} x_i))_1^l$, then there exists $i \geq 1$ such that $T_i(a_2) > (x_i)_1^l$; set $t_2 = T_i(a_2)$.

Now, the result is true for each $\alpha' < \alpha$, and $o(R(t_2)) > \omega^{\alpha'} \cdot k$ for each $\alpha' < \alpha$ and every k, so there exists $F_2 \in \mathcal{F}(R(t_2))$, $m_2 > \max F_1$, and $(a_j)_{F_2}$ such that $\sum_{F_2} a_j = 1$ and every subset G of F_2 which is also in $\mathcal{S}_{\alpha_{m_2}}$ satisfies $\sum_G a_j < \varepsilon_2$, where ε_2 was chosen to be less than $1/n$. Now, by [OTW], there exists m such that if $G \geq m$ and $G \in \mathcal{S}_{\alpha_i}$ for any $i < m_2$, then $G \in \mathcal{S}_{\alpha_{m_2}}$. Also, since ω^α is a limit ordinal, we may remove a finite number of the smallest nodes of $R(t_2)$ without changing the order of the tree and so we may choose $F_2 \geq m$.

We continue in this fashion, as before, to obtain $F_1 < \cdots < F_n$, $(a_j)_{F_l}$ such that if $i \leq \max F_{l-1}$, $G \in \mathcal{S}_{\alpha_i}$, $G \subseteq F_l$, then $\sum_G a_j < \varepsilon_l < 1/n$. Set $F = \bigcup F_l$ and $\overline{a}_j = \frac{1}{n} a_j$ for $j \in F$. Let $G \in \mathcal{S}_\alpha$, then there exists $j \geq 1$ such that $G \in \mathcal{S}_{\alpha_j}$ and $j \leq G$. As before let i be least such that $G \cap F_i \neq \emptyset$, then $j < m_l$ $(l > i)$ and so $\sum_{G\cap F_l} \overline{a}_j = \frac{1}{n}\sum_{G\cap F_l} a_j < \varepsilon_l/n$ and $\sum_{G\cap F_i} \overline{a}_j \leq \sum_{F_i} \overline{a}_j = 1/n$. Thus

$$\sum_G \overline{a}_j < \frac{1}{n}(1+\varepsilon_i+\cdots+\varepsilon_n) < \frac{2}{n} < \varepsilon$$

giving the required contradiction.

The case for $m > 1$ proceeds along similar lines as for the successor case; we just need to pick n so that $m/n < \varepsilon/2$. This completes the proof of the proposition. ∎

Proof of Theorem 5.19: We first note that for each $n \geq 1$, if $E \in [\mathcal{S}_\alpha]^n$, then $\|\sum_{i\in E} a_i e_i\| \geq 2^{-n}\sum |a_i|$, from the definition of the norm on $T(\mathcal{S}_\alpha, 1/2)$, thus we may construct a block basis tree isomorphic to Tree($[\mathcal{S}_\alpha]^n$). As we noted in Definition 5.16, $o(\text{Tree}([\mathcal{S}_\alpha]^n)) = \omega^{\alpha \cdot n}$, and hence $I_b(T(\mathcal{S}_\alpha, 1/2)) > \omega^{\alpha \cdot n}$ for each $n \geq 1$, and so $I_b(T(\mathcal{S}_\alpha, 1/2)) \geq \omega^{\alpha \cdot \omega}$.

Now, suppose $I_b(T(\mathcal{S}_\alpha, 1/2)) > \omega^{\alpha \cdot \omega}$, then there exists an ℓ_1-K-block basis tree T of order $\omega^{\alpha \cdot \omega}$ and by Fact 5.3 (v) we may write $\omega^{\alpha \cdot \omega} = \omega^{\omega^{\theta+1}}$ for some $\theta < \omega_1$. This is one of the fixed points of our construction by Remark 4.3 (ii). Thus for every $\varepsilon > 0$ there exists an ℓ_1-block tree of T with constant $1+\varepsilon$ and order $\omega^{\alpha \cdot \omega}$, so we may assume T has constant $1+\varepsilon$ where $\varepsilon < 1/10$.

Let $m = 1$ and choose n from Proposition 5.20. Since $o(T) > \omega^\alpha \cdot n$ there exist $F \in \mathcal{F}(T)$, $F = \{n_1, \ldots, n_l\} = (\min \operatorname{supp} x_i)_1^l$ for some $(x_i)_1^l \in T$ and $(a_j)_F \subset \mathbf{R}^+$ such that $\sum_F a_j = 1$ and $\sum_G a_j < \varepsilon/3$ for each subset $G \subseteq F$ which is also in $\mathcal{S}_\alpha$; set $x = \sum_{i=1}^l a_{n_i} x_i$. To calculate the norm of x let $(E_i)_1^k$ be $\mathcal{S}_\alpha$ admissible. Let $I = \{i\colon \operatorname{supp}(x_i) \subseteq E_j \text{ for some } j\}$, let

$$J = \{i \leq l\colon i \notin I \text{ and } \operatorname{supp}(x_i) \cap E_j \neq \emptyset \text{ for some } j\}$$

and note that since $(E_i)_1^k$ is $\mathcal{S}_\alpha$ admissible, there exist $A, B, C \in \mathcal{S}_\alpha$ such that $\{n_j\colon j \in J\} = A \cup B \cup C$. Now

$$\begin{aligned}
\frac{1}{2}\sum_{j=1}^k \|E_j x\| &\leq \frac{1}{2}\sum_{i=1}^l a_{n_i} \sum_{j=1}^k \|E_j x_i\| \\
&= \frac{1}{2}\left(\sum_{i \in I} a_{n_i} \sum_{j=1}^k \|E_j x_i\| + \sum_{i \in J} a_{n_i} \sum_{1}^k \|E_j x_i\|\right) \\
&\leq \frac{1}{2}\sum_{i \in I} a_{n_i} + \sum_{i \in J} a_{n_i} \frac{1}{2}\sum_{j=1}^k \|E_j x_i\| \\
&\leq \frac{1}{2} + \sum_{i \in J} a_{n_i} \|x_i\| \\
&\leq \frac{1}{2} + \sum_{j \in A \cup B \cup C} a_j \\
&\leq \frac{1}{2} + 3\frac{\varepsilon}{3}
\end{aligned}$$

and hence $\|x\| \leq 1/2 + \varepsilon$. However $(x_i)_1^l \in T$, an ℓ_1-$(1+\varepsilon)$-tree, and so $\|x\| \geq 1/(1+\varepsilon)$, a contradiction. Thus $I_b(T(\mathcal{S}_\alpha, 1/2)) = \omega^{\alpha \cdot \omega}$. ∎

Remark 5.21: The authors have recently calculated the index of two other classes of Banach spaces. In [JO] it is shown that the index for $C(K)$, where K is a countable compact metric space, is given by

$$I(C(\omega^{\omega^\alpha})) = \omega^{1+\alpha+1}, \quad I_b(C(\omega^{\omega^\alpha})) = \omega^{\alpha+1} \quad (1 \le \alpha < \omega_1)$$

and for X_α the αth Schreier space, $1 \le \alpha < \omega_1$ (Definition 5.18), that $I(X_\alpha) = \omega^{\alpha+1}$.

6. Final remarks

As we noted in the introduction, Theorem 1.1 is false for $1 < p < \infty$. This is a consequence of ℓ_p being arbitrarily distortable [OS]. In particular the following is true.

THEOREM 6.1: *For each p, $1 < p < \infty$, and every $L \ge 1$, there exist $K > 1$ and $\alpha < \omega_1$ such that for any $\beta < \omega_1$ there exists a Banach space X which contains an ℓ_p-K-tree on X of order at least β, but no ℓ_p-L-tree of order α.*

Proof: Fix $L \ge 1$; then since ℓ_p is arbitrarily distortable there exists a Banach space X isomorphic to ℓ_p satisfying $d(Y, \ell_p) > 2L$ for every subspace Y of X. Clearly, as X is isomorphic to ℓ_p, there exists some constant K so that X contains an ℓ_p-K-tree on X of order β for each $\beta < \omega_1$. If the theorem is false, then for each $\alpha < \omega_1$ there would exist an ℓ_p-L-tree on X of order at least α. This in turn would imply [B] that X contains a subspace Y with $d(Y, \ell_p) \le L$, contradicting our original assumption. This completes the proof. ∎

The finite version of Theorem 1.1 for ℓ_p is true, as we mentioned in the introduction. From this and our construction of T_ω (Definition 3.1) it is easy to see that if we have an ℓ_p-K-tree T of order ω on a Banach space X, then there exists a block tree of T which is an ℓ_p-$(1+\varepsilon)$-tree of order ω. Thus it seems reasonable to ask the following question.

QUESTION 2: *For which ordinals α is Theorem 1.1 true for $1 < p < \infty$, and what is their supremum?*

Definition 6.2: We extend the definition of the ℓ_1-spreading models introduced by Kiriakouli and Negrepontis [KN] to ℓ_p $(1 \le p \le \infty)$. A sequence $(x_n)_{n=1}^\infty$ has an **ℓ_p-$\mathcal{S}_\alpha$-spreading model ($\mathcal{S}_\alpha$-SM**, for some $1 \le p \le \infty$, with constant K, if $(x_i)_{i\in F} \overset{K}{\sim}$ uvb $\ell_p^{|F|}$ for every $F \in \mathcal{S}_\alpha$, where $\mathcal{S}_\alpha$ is the collection of Schreier sets of order α introduced in Section 5.

We can refine the constant of an ℓ_1-SM from K to $(1+\varepsilon)$ on a block basis as we did above for ℓ_1-trees, but the proof is much more straightforward. We also note that these spreading models are a stronger notion than ℓ_1-trees.

We need the following result [OTW]:

LEMMA 6.3 ([OTW]): *For each pair $\alpha, \beta < \omega_1$ there exists $N \subseteq \mathbf{N}$ such that $\mathcal{S}_\alpha[\mathcal{S}_\beta](N) \subseteq \mathcal{S}_{\beta+\alpha}$.*

THEOREM 6.4: *For any $K > 1$, every $\varepsilon > 0$, and each $\alpha < \omega_1$, there exists $\beta < \omega_1$ such that if (x_n) is a normalized basic sequence having an ℓ_1-$\mathcal{S}_\beta$-SM with constant K, then there exists a normalized block basis (y_n) of (x_n) having an ℓ_1-$\mathcal{S}_\alpha$-SM with constant $1 + \varepsilon$.*

Proof: This follows immediately from the following lemma. ∎

LEMMA 6.5: *Let (x_n) be a normalized basic sequence having an ℓ_1-$\mathcal{S}_{\alpha\cdot 2}$-SM with constant K. Then there exists a normalized block basis (y_n) of (x_n) having an ℓ_1-$\mathcal{S}_\alpha$-SM with constant $\sqrt{K}$.*

Proof: For fixed $\alpha < \omega_1$ choose, by Lemma 6.3, $N = (n_i) \subseteq \mathbf{N}$ such that $\mathcal{S}_\alpha[\mathcal{S}_\alpha](N) \subseteq \mathcal{S}_{\alpha\cdot 2}$ and consider the subsequence $(x_{n_i})_1^\infty$. We know that since $\mathcal{S}_{\alpha\cdot 2}(N) \subseteq \mathcal{S}_{\alpha\cdot 2}$,

$$\Big\|\sum_{i\in F} a_i x_{n_i}\Big\| \geq \frac{1}{K}\sum_{i\in F}|a_i|, \qquad \text{for every } (a_i) \subset \mathbf{R}, \text{ and } F \in \mathcal{S}_{\alpha\cdot 2}.$$

If there exists $k \geq 1$ such that

$$\Big\|\sum_{i\in E} a_i x_{n_i}\Big\| \geq \frac{1}{\sqrt{K}}\sum_{i\in E}|a_i|, \qquad \text{for every } (a_i) \subset \mathbf{R}, \text{ and each } E \in \mathcal{S}_\alpha \text{ with } E > k$$

then we are finished since $E \in \mathcal{S}_\alpha$ implies $E + k \in \mathcal{S}_\alpha$ $(k \geq 1)$.

Otherwise there exists a normalized block basis (y_j) of (x_{n_i}) satisfying

$$y_j = \sum_{i\in E_j} a_i x_{n_i}, \ \sum_{i\in E_j}|a_i| > \sqrt{K}$$

with $E_j \in \mathcal{S}_\alpha$ and $E_j < E_{j+1}$ for each $j \geq 1$. Now, for each $E \in \mathcal{S}_\alpha$ the set $F = \bigcup_{j\in E} E_j$ is an element of $\mathcal{S}_\alpha[\mathcal{S}_\alpha](N)$, which in turn is contained in $\mathcal{S}_{\alpha\cdot 2}$. Thus we obtain

$$\Big\|\sum_E b_j y_j\Big\| \geq \frac{1}{\sqrt{K}}\sum_E |b_j|, \qquad \text{for every } (b_j) \subset \mathbf{R}, \text{ and } E \in \mathcal{S}_\alpha$$

using James' argument as in the proof of Theorem 1.1. ∎

Remark 6.6: We note here some closing points for this section.

(i) For every $\alpha < \omega_1$ there exists a Banach space X_α with an ℓ_1-tree of order α but X_α has no ℓ_1-spreading models. In fact X_α can be taken to be reflexive with all normalized weakly null sequences having an ℓ_2-$(1+\varepsilon)$ subsequence.

Proof: We use a similar construction to Szlenk [Sz]. Let $X_k = \ell_1^k$ $(k \geq 1)$. If $\alpha < \omega_1$ is a limit ordinal and we have constructed X_β for each $\beta < \alpha$, let $X_\alpha = (\sum_{\beta<\alpha} X_\beta)_{\ell_2}$. Given X_α, let $X_{\alpha+1} = (X_\alpha \oplus \mathbf{R})_{\ell_1}$. ∎

(ii) As for the ℓ_p-trees, Theorem 6.4 is also true for $p = \infty$ and false for $1 < p < \infty$. This follows from the proof of Theorem 6.1.

(iii) It follows from Lemma 6.5 that if (x_n) is a normalized basic sequence having an ℓ_1-$\mathcal{S}_{\omega^\alpha}$-SM with any constant, then for every $\beta < \omega^\alpha$, and any $\varepsilon > 0$, there exists a normalized block basis (y_n) of (x_n) having an ℓ_1-$\mathcal{S}_\beta$-SM with constant $1 + \varepsilon$.

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